STABILITY OF LOW-VOLTAGE ARC WITH NONEQUILIBRIUM IONIZATION

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A stability criterion is obtained for a low-voltage arc in the absence of ionization equilibrium. It is shown that the stability depends on the properties of the generation function. The example of a cesium arc of a thermionic converter illustrates the influence of a change of the electron temperature on the stability of a plasma.

1. The aim of this paper is to investigate the stability of a low-voltage arc with nonequilibrium generation function in a practically important example – a cesium thermionic converter. The generation function may depart from equilibrium under the conditions of a cesium arc for many reasons – depletion of the "tail" of the Maxwellian distribution, the presence of beam electrons, escape of radiation, diffusion of excited atoms. The influence of each of these factors separately or together on the form of the function has been studied on many occasions, since the generation function is important for the design of the arc and the construction of the current–voltage characteristic of the converter.

Many people have noted that to construct the current–voltage characteristic it is important to know the integral properties of the generation function, whereas the more subtle properties of this function are important for the stability.

Without restricting ourselves to any particular form of the generation function, we formulate the problem as follows: What properties must the generation function have for the system to be unstable?

2. If the gradients of the electron temperature can be ignored, a low-voltage arc in cesium vapor is described by the equation of ambipolar diffusion and the equation of electron heat conduction [1]:

$$-D_a d^2n / dx^2 = \alpha^2 (T_e) F(n) \tag{2.1}$$

$$jV = 2T_c j_R (T_e / T_c - 1) + I v_i n_1 \tag{2.2}$$

where D_a is the coefficient of ambipolar diffusion, n is the plasma density, j is the current density, V is the potential difference across the gap, I is the effective ionization potential, T_c is the cathode temperature, x is the coordinate perpendicular to the surface of the electrodes, F (n) is a nonmonotonic function of the plasma density, α^2 (T_e) is the generation function as a function of the temperature of the electrons, and v_i is the thermal velocity of the ions.

The function F (n, T_e) in the general case cannot be represented as a product, though, as we shall see below, in this case this is not important and is introduced for clarity. Allowance for the escape of radiation in the energy-balance equation does not lead to qualitatively new results and, therefore, in what follows it will be ignored for simplicity. To (2.1) and (2.2) one must add the boundary conditions for a developed arc [1]:

$$j = j_R - \frac{n_1 v_e}{4} \exp\left(-\frac{\varphi_c}{T_e}\right), \quad D_a \frac{dn}{dx}\bigg|_{x=-d} = \frac{n_1 v_i}{2} \tag{2.3}$$

$$j = \frac{n_2 v_e}{4} \exp\left(-\frac{\varphi_a}{T_e}\right), \quad -D_a \frac{dn}{dx}\bigg|_{x=d} = \frac{n_2 v_i}{2} \tag{2.4}$$

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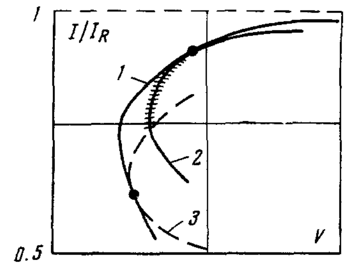

Fig. 1

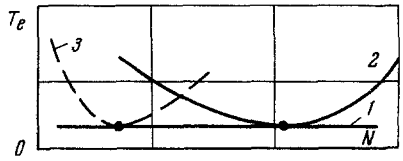

Fig. 2

where n_1 and n_2 are the plasma density at the cathode and the anode; φ_c and φ_a are the discontinuities of the potential at the electrodes; v_e is the thermal velocity of the electrons; and 2d is the distance between the cathode and the anode (the gap). Because of the symmetry of the boundary conditions (2.3) and (2.4), $n_1 = n_2$.

Equation (2.1) is similar to the equation of motion of a particle in a potential well with energy $u = \int_0^n F(n)\, dn$, the role of the particle coordinate being played by n and the role of the time by x. To within a term of order l/d, the plasma density on the boundary can be set equal to zero (l is the electron mean free path). Then the problem consists of finding a point $N = n\ (x=0)$ such that if the particle begins its motion at this point with zero velocity [because of the symmetry of the solution $(dn/dx)_{x=0} = 0$] it arrives at the point with coordinate $n = 0$ in a "time" equal to a given value (half the gap d). Integrating (2.1), we obtain

$$\frac{d\alpha}{\sqrt{D_a}} = \tau(N) = \frac{1}{\sqrt{2}} \int_0^N \frac{dn}{\sqrt{u(N) - u(n)}} \tag{2.5}$$

We shall call τ (N) the period. Going over in (2.5) to integration over u, we can write

$$\tau(E) = \frac{1}{\sqrt{2}} \int_0^E \frac{(dn/du)\, du}{\sqrt{E-u}} \tag{2.6}$$

where $E = u(N)$ is the "total energy" of the particle.

Depending on the form of the potential u (n), τ is a function of the amplitude N [or $\tau(E)$] may be nonmonotonic [2], i.e., to one value of τ, or α, there may correspond two or more values of N. It is necessary to establish what properties the "force" F must have if τ (N) is to be a nonmonotonic function. For this, using the integral Abel equation (2.6) we must express u (n) in terms of τ (N) [3]:

$$n(u) = \frac{1}{\sqrt{2}\pi} \int_0^u \frac{\tau(E)\, dE}{\sqrt{u-E}} \tag{2.7}$$

Since $F \geq 0$, the dependence of u on n [or E (N)] is monotonic – u and n are related one-to-one; knowing n (u) we can find u (n). Specifying different nonmonotonic dependences τ (N) one can show that nonunique solutions (2.1) can also hold for smooth functions F (n), and F (n) may even change curvature.

The multivaluedness of the solutions depends on subtle properties of the generation function, and therefore an approximate method used to determine the ambiguity of the solution from the form of the function F (n), used, for example, in [2, 4], is restricted.

If there are two or more solutions, the solutions for which $\partial\tau/\partial N < 0$ are unstable and the solutions with $\partial\tau/\partial N > 0$ are stable. This will be proved below under the assumption that the electron temperature does not change.

3. To determine the instability region as a function of the external parameters (current, voltage, etc.), it is necessary to construct the current–voltage characteristic of the arc. The potential difference across the gap is composed of the potential difference across the plasma, φ_p, and the difference of the electrode discontinuities of the potential. To within a quantity of order of l/d, the latter can be found from the boundary conditions with allowance for $n_1 = n_2$:

$$\varphi_c - \varphi_a = T_e \ln j / j_R (1 - j / j_R) \tag{3.1}$$

The potential drop across the plasma can be found by integrating the equation for the electron current:

$$j_e = -D_e \, dn / dx + u_e n d\varphi / dx \tag{3.2}$$

where u_e is the electron mobility and φ is the potential of the plasma.

Since the variation of the electron current over the gap is small, with allowance for (2.5) we obtain

$$\varphi_p = \frac{j}{u_e} \int \frac{dx}{n(x)} = \frac{\sqrt{2D_a}}{\alpha(N)} \frac{j}{u_e} \int_{n_1}^{N} \frac{dn}{n\,[u(N) - u(n)]^{1/2}} \tag{3.3}$$

The value of n_1 can be found from (2.3) by noticing that this is the "velocity" of the particle at the end of the trajectory:

$$n_1(N) = 2\sqrt{2D_a u(N)}\,\alpha(N) / v_i \tag{3.4}$$

Using (3.4) we must eliminate n_1 from (3.3) and (2.2). The total voltage across the gap can be obtained from (3.1) and (3.3):

$$V = \varphi_c - \varphi_a + \varphi_p \tag{3.5}$$

For the two unknowns V and N (we assume the current is given) we have the two equations (2.2) and (3.5). These equations constitute the current–voltage characteristic in parametric form (N is the parameter). If $F \sim n$, then all the integrals can be expressed in elementary functions and the current–voltage characteristic is obtained with a section of negative differential resistance (see curve 1 in Fig. 1) [1]. As follows from (2.5), the dependence T_e (N) is a straight line parallel to the abscissa (see curve 1 in Fig. 2). If T_e depends nonmonotonically on N (the curves 2 and 3 in Fig. 2) the nonmonotonicity of V as a function of j is expressed to a greater degree than when T_e =const (curves 2 and 3 in Fig. 1).

Although α may vary appreciably, T_e varies little, since α usually depends on the electron temperature exponentially:

$$\alpha^2 = \alpha_0^2 e^{-I/T_e} \tag{3.6}$$

At low currents through the device, N is small. Almost all the potential difference occurs across the plasma. This section of the current–voltage characteristic can be obtained from (2.2). Since N is small, the ionization losses can be ignored, and therefore $V = 2T_c\,(T_e/T_c - 1)\, j_R/j$ and the voltage decreases with increasing current, $dV/dj < 0$. If T_e (N) is nonmonotonic, the temperature decreases with increasing N (at low plasma densities and therefore at low currents). This means that dV/dj becomes even smaller. With increasing N, the potential difference across the plasma, φ_p, decreases and the main losses of energy from the plasma are ionization losses:

$$jV = 2\sqrt{2D_a u(N)}\, I\alpha(N) \tag{3.7}$$

Multiplying (3.1) by j and ignoring φ_p, we obtain

$$2\sqrt{2D_a u(N)}\, I\alpha(N) = T_e j \ln j / j_R (1 - j / j_R) \tag{3.8}$$

It can be seen from (3.8) that with increasing N the current density increases, and then so does V in accordance with (3.1). Since T_e increases with increasing N at high densities (see Fig. 2), the derivative dV/dj is greater than in the case T_e =const. These conclusions agree qualitatively with the calculations of a thermionic converter on a computer made in [5], where it was shown that when allowance is made for the escape of resonance radiation and the diffusion of excited atoms the negative section of the current–voltage characteristic is appreciably increased. Allowance for the escape of resonance radiation and diffusion of excited atoms means that the generation function at low plasma densities is proportional to n^2. In this case α (and, accordingly, T_e) is a nonmonotonic function of N, as was assumed above. For some discharge parameters, the point $\partial\tau/\partial N = 0$ may lie on a section of the current–voltage characteristic with $dj/dV > 0$. This means that the instability can also occur on a positive section of the characteristic (see curve 2 in Fig. 1).

4. The instability condition is derived with neglect of perturbations of the electron temperature. We now obtain conditions under which this is justified.

Varying (2.3) and (2.4), we find

$$\frac{\delta j}{j}\frac{j_R}{j_R-j}=\frac{\delta n_2-\delta n_1}{n_1}-\frac{\delta T}{T_e}\frac{V}{T_e}+\frac{\delta V}{T_e} \tag{4.1}$$

$$D_a\frac{d}{dx}\delta n\,|_{x=-d}=\frac{v_i}{2}\delta n_1,\quad -D_a\frac{d}{dx}\delta n\,|_{x=d}=\frac{v_i}{2}\delta n_2 \tag{4.2}$$

Let us consider the case of high currents, which is clearly of most interest. In this case, the potential difference across the plasma can be ignored. The nonstationary equation for the perturbations of the electron temperature has the form

$$V\delta j+j\delta V=2j_R\delta T+{}^1/_2 Iv_i\,(\delta n_1+\delta n_2)+2\varkappa k_y^2 d\delta T-3\Gamma dN\delta T \tag{4.3}$$

where $\varkappa$ is the coefficient of electron thermal conductivity.

In (4.3) we consider perturbations of the plasma parameters along the electrodes with wavelength $\lambda=2\pi/k_y$, their time dependence being represented in the form $\exp(-\Gamma t)$.

Together with the nonstationary equation of ambipolar diffusion for the perturbations

$$-\Gamma\delta n+D_a k_y^2\delta n-D_a\frac{d^2}{dx^2}\delta n=\alpha^2\left.\frac{\partial F}{\partial n}\right|_{n=n_0(x)}\delta n+F\frac{\partial\alpha^2}{\partial T_e}\delta T \tag{4.4}$$

Eqs. (4.1)-(4.3) form a closed system of homogeneous equations, from which the decay rate Γ can be determined.

It is not possible to obtain a solution of (4.4) in the general case. We must therefore consider how the stability is affected by the term with δT in (4.4), whose properties without this term will be investigated below.

For the same reasons as in the stationary case

$$\delta n_1=\delta n_2 \tag{4.5}$$

This enables us to omit $(\delta n_1-\delta n_2)$ in (4.1). Depending on the load resistance, one must consider several cases.

The case $\delta V=0$ can occur when in the external circuit the voltage is given or when, for given current, one has perturbations along the electrodes, $k_y\neq 0$. Suppose $k_y\to 0$.

As follows from (4.1), the current and the temperature vary in antiphase. With allowance for (4.2), we obtain the connection between the perturbation of the temperature and the density δn_1 on the boundary:

$$\frac{\delta T}{T_e}=-\frac{Iv_i n_1}{V^2 j\,(j_R-j)/T_e j_R+2j_R T_e}\frac{\delta n_1}{n_1} \tag{4.6}$$

Substituting (4.6) into (4.4), we find that the perturbation of the temperature in the given case is a stabilizing influence. For currents $j<j_R$ this influence can be ignored if

$$\frac{I}{V}\sqrt{\frac{m}{M}}e^{\varphi_a/T_e}\ll 1 \tag{4.7}$$

For large currents ($j\approx j_R$) the second term on the right side of (4.4) is greater than the first and the ratio of these terms is of order

$$\frac{I}{T_e}\sqrt{\frac{m}{M}}e^{\varphi_a/T_e}>1 \tag{4.8}$$

In this case the state of the plasma is stable.

If $\delta j=0$ and $k_y=0$, a variation in the external circuit increases the temperature:

$$\frac{\delta V}{T_e}=\frac{V}{T_e}\frac{\delta T}{T_e}$$

and at currents near j_R an increase of the temperature increases the plasma density:

$$\frac{\delta T}{T_e}=\frac{Iv_i n_1}{jV}\frac{\delta n_1}{n_1}$$

Such a dependence enhances the instability.

If $\delta V=0$, $k_y \neq 0$, $j \approx j_R$, the high electron thermal conductivity will equalize the perturbation of the electron temperature, reducing the stabilizing influence. For this, k_y must be sufficiently large:

$$\left(\frac{I}{T_e}\right)^2 \sqrt{\frac{m}{M}} \ll 4k_y^2 d^2$$

On the other hand, waves that are too short would be damped because of diffusion of particles along the electrodes. If this is not to happen, we require

$$k_y^2 d^2 \ll d^2 \Gamma / D_a$$

Thus

$$(I / T_e)^2 \sqrt{m / M} \ll 4k_y^2 d^2 \ll 4d^2 \Gamma / D_a \tag{4.9}$$

If these conditions are satisfied, the term with Γ in (4.3) is small and the variation of the electron temperature can be ignored. One can estimate Γ by setting it equal in order of magnitude to the derivative of the rate of ionization, since the process of nonequilibrium ionization is responsible for the instability. For the characteristic conditions of operation of the converter considered, for example, in [5] ($P_{Cs}=1$ mm Hg, $j_R=0.66$ A, $d=0.5$ mm, $T_e=3000$°K) the rate of ionization is proportional to n^2 and therefore $4\Gamma d^2/D_a \gg 1$. The condition (4.9) can be satisfied and in the system an instability can develop with characteristic wavelengths exceeding the gap width by an order of magnitude.

5. We prove that the decay rate Γ has the same sign as the derivative $\partial\tau/\partial N$. The equation for the perturbation of the plasma density in the gap without allowance for the perturbation of the temperature is

$$-\Gamma \delta n - D_a \frac{d^2}{dx^2} \delta n = \alpha^2 \frac{\partial F}{\partial n}\Big|_{n=n_0(x)} \delta n(x) \tag{5.1}$$

In (5.1) the time dependence is chosen in the form $\exp(-\Gamma t)$ and n_0 is the stationary solution. We introduce the new notation

$$\frac{d^2\Gamma}{D_a} = E, \quad -\frac{d^2\alpha^2}{D_a}\frac{\partial F}{\partial n}\Big|_{n=n_0(x)} = U(x), \quad \delta n \equiv \psi(x) \tag{5.2}$$

With allowance for this, (5.1) can be written in the form

$$-\psi'' + U(x)\psi = E\psi \tag{5.3}$$

with the boundary conditions $\psi(-1)=\psi(1)=0$.

In (5.3) the coordinate x is made dimensionless by division by d. This equation has the form of a Schrödinger equation for a particle with potential U (x). Depending on the form of the stationary solution n (x), i.e., on the form and depth of the potential well U (x), the energy levels may be negative or positive. If the lowest level E_0 is negative, the system is unstable; if positive, stable, since Γ and E have the same sign. The properties of this ground state will be investigated subsequently. We show first that the sign of the ground level is the same as the sign of the function φ (x=1), which is a solution of the equation

$$-\varphi'' + U(x)\varphi = 0 \tag{5.4}$$

with the boundary conditions

$$\varphi(-1) = 0, \quad \frac{d\varphi}{dx}\Big|_{x=-1} > 0$$

(Cauchy problem).

For $E_0=0$, the solutions of Eqs. (5.3) and (5.4) are the same, and then $\varphi(1)=0$. If $\varphi(1) < 0$, there is a point $x_0 < 1$ such that $\varphi(x_0)=0$. Since the number of zeros on the interval $(-1, 1)$ is a monotonic function of the energy [6], and the energy corresponding to the function $\varphi(x)$ is zero, the energy E_0 corresponding to the function $\psi(x)$, which has no zeros on the given interval, is less than zero. If $\varphi(1) > 0$, then because $\varphi(x)$ in this case has no zeros (see below) on the interval $(-1, 1)$, the point x_0 lies to the right of unity. Since $\psi(x)$ vanishes over the extended interval $(-1, x_0)$ (at the point x=1), the energy corresponding to it is greater than zero.

The sign of the function $\varphi(x)$ for x=1 coincides with the sign of the energy level of the eigenvalue problem (5.3). It is necessary to calculate the value of this function at x=1.

One solution of Eq. (5.4) is known:

$$\varphi_1 = dn_0/dx \tag{5.5}$$

This can be seen by substituting (5.5) in (5.3) and comparing the resulting equation with the differentiated Eq. (2.1).

We find a second solution φ_2 (x) of (5.4) from the condition of conservation of the Wronskian:

$$\varphi_1'\varphi_2 - \varphi_2'\varphi_1 = C_2$$

We obtain

$$\varphi(x) = C_1\varphi_1 + C_2\varphi_1 \int_{-1}^{x} \frac{dx}{\varphi_1^2(x)} \tag{5.6}$$

Since

$$\varphi_1(-1) = \left[(2\alpha^2/D_a)\int_0^N F\,dn\right]^{1/2} > 0$$

we find $C_1 = 0$ from the condition $\varphi(-1) = 0$. From $C_1 = 0$ we find $(d\varphi/dx)_{x=-1} > 0$. Note here that, by the alternation of zeros theorem [6], the function φ has no zeros on (0, −1).

The solution of (5.6) with $C_1 = 0$ does not apply for $x > 0$, since when $x = 0$ the integral diverges on account of $\varphi_1(0) = 0$. Therefore, for $x > 0$ the solution of (5.4) can be written in the form

$$\varphi(x) = C_1'\varphi_1(x) + C_2'\varphi_1 \int_{x}^{1} \frac{dx}{\varphi_1^2} \tag{5.7}$$

The constants C_1' and C_2' are found from the conditions of continuity of the functions (5.6) and (5.7) and their derivatives for $x = 0$:

$$\left[C_2\varphi_1(x)\int_{-1}^{x}\frac{dx}{\varphi_1^2}\right]_{x\to-0} = \left[C_1'\varphi_1(x) + C_2'\varphi_1(x)\int_{x}^{1}\frac{dx}{\varphi_1^2}\right]_{x\to+0} \tag{5.8}$$

$$\left\{C_2\frac{d}{dx}\left[\varphi_1(x)\int_{-1}^{x}\frac{dx}{\varphi_1^2}\right]\right\}_{x\to-0} = \left\{C_1'\varphi_1(x) + C_2'\frac{d}{dx}\left[\varphi_1\int_{x}^{1}\frac{dx}{\varphi_1^2}\right]\right\}_{x\to+0} \tag{5.9}$$

Noting $\varphi_1(0) = 0$ and $\varphi_1(-x) = -\varphi_1(x)$, we find from (5.8) that $-C_2' = C_2$. From (5.9) we obtain

$$C_1' = \left\{\frac{2C_2}{\varphi_1'(0)}\frac{d}{dx}\left[\varphi_1(x)\int_{-1}^{x}\frac{dx}{\varphi_1^2}\right]\right\}_{x\to-0}$$

Noting from (5.7) that $\varphi(1) = C_1'\varphi_1(1)$, we obtain

$$\varphi(1) = 2\frac{\varphi_1(1)}{\varphi_1'(0)}\left\{C_2\frac{d}{dx}\left[\varphi_1(x)\int_{-1}^{x}\frac{dx}{\varphi_1^2}\right]\right\}_{x\to-0} \tag{5.10}$$

Since

$$\varphi_1(1) < 0,\ \varphi_1'(0) = d^2n/dx^2 \sim -F(n) < 0,\ C_2 > 0$$

the sign of φ (1) coincides with the sign of the derivative of the function in the square brackets. Since $C_1' < 0$ and $C_2' < 0$ for $\varphi(1) > 0$, it follows by the alternation of zeros theorem that φ (x) also has no zeros on (0.1). We show that the derivative (5.10) is equal to $\partial\tau/\partial N$. The condition $x \to 0$ corresponds to $n \to N$. With allowance for this and going over in $\left\{d\left[\varphi_1\int_{-1}^{x}dx/\varphi_1^2\right]\Big/dx\right\}_{x\to 0}$ from the variable x to the variable n, we can show that, to within a positive factor $\alpha/\sqrt{D_a}$, this expression is equal to $\partial\tau/\partial N$. The decay rate Γ and the derivative $\partial\tau/\partial N$ have the same sign.

In conclusion, we should point out that similar questions relating to the stability of the solutions of nonlinear equations in different physical problems have already been considered in [7, 8].

The paper [7] is completely devoted to the case of an unbounded interval $(-\infty, \infty)$, while in [8] only one special form of the generation function was considered.

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